Name: ______________________________ Student Number: _________________________

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HSC ASSESSMENT TASK 3

TERM 2 2016

Mathematics Extension 1

Date: Tuesday, 14 June, P1

Time: 60 minutes plus 2 minutes reading time

General Instructions:

- Reading time – 2 minutes
- Working time – 60 minutes
- Write using black pen
- Board approved calculators may be used
- A Reference Sheet is provided
- In Questions 6-7, show relevant mathematical reasoning and/or calculations

Total marks: 35

Section I

5 marks

Attempt Questions 1-5

Allow about 8 minutes for this section

Section II

30 marks

Attempt Questions 6-7

Allow about 52 minutes for this section

Outcomes to be assessed are:

A student:

HE3 uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay.

HE4 uses the relationship between functions, inverses functions and their derivatives.

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1-5.

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1 Which of the following equates to $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$?

(A) $-\frac{\sqrt{3}}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\sqrt{3}}{2}$

(D) $-\frac{1}{2}$

2 The velocity v of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 60 + 8x - 4x^2$. What is the centre of the motion?

(A) $x = 1$

(B) $x = 3$

(C) $x = -5$

(D) $x = 60$

3 What is the domain and range of $y = 2\cos^{-1}(x-1)$?

(A) Domain: $0 \le x \le 2$ and Range: $0 \le y \le \pi$

(B) Domain: $-1 \le x \le 1$ and Range: $0 \le y \le \pi$

(C) Domain: $0 \le x \le 2$ and Range: $0 \le y \le 2\pi$

(D) Domain: $-1 \le x \le 1$ and Range: $0 \le y \le 2\pi$

4 A particle moves with velocity $v = \sqrt{9 - x^2}$. If initially the particle has displacement $x = 3$, which of the following is the displacement equation?

(A) $x = \cos\left(t - \frac{\pi}{2}\right) + 3$

(B) $x = 3\sin\left(t + \frac{\pi}{2}\right)$

(C) $x = 2 - \sin\left(t - \frac{\pi}{2}\right)$

(D) $x = 3 - \cos\left(t + \frac{\pi}{2}\right)$

5 A stone is thrown at an angle of α to the horizontal. The position of the stone after t seconds is given by the equations $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ where g m/s^2 is the acceleration due to gravity and V m/s is the initial velocity of projection.

What is the maximum height reached by the stone?

(A) $\dfrac{V\sin\alpha}{g}$

(B) $\dfrac{g\sin\alpha}{V}$

(C) $\dfrac{V^2\sin^2\alpha}{2g}$

(D) $\dfrac{g\sin^2\alpha}{2V^2}$

Section II

30 marks
Attempt Questions 6-7
Allow about 52 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6-7, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The volume, V, of a sphere of radius r millimetres is increasing at a constant rate of 160 mm^3 per second. The volume of a sphere can be calculated using the formula $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere is $A = 4\pi r^2$.

(i) Find $\frac{dr}{dt}$ in terms of r. **2**

(ii) Find the rate of change of the surface area A of the sphere when the radius is 40 mm. **2**

(b) Find $\int \frac{1}{\sqrt{9-4x^2}}\,dx$ **2**

(c) A particle is moving in simple harmonic motion about the origin O such that its velocity v ms^{-1} satisfies $v^2 = 9(4-x^2)$, where x is the displacement of the particle from O. The initial velocity of the particle is zero. How many seconds will it take the particle to first reach O? **3**

(d) The velocity of a particle moving in a straight line is given by $v = 10 - x$ where x metres is the distance from a fixed point O and v is the velocity in metres per second. Initially the particle is at the origin, O.

(i) Find an expression for the acceleration. **2**

(ii) Show that $x = 10 - 10e^{-t}$ by integration. **3**

(iii) What is the limiting position of the particle? **1**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A rock is projected horizontally from the top of a 25 metre high cliff. The rock is thrown with an initial velocity of 40 ms^{-1}. Assume $g = 10 \text{ ms}^{-2}$.

(i) Show that the parametric equations of the path are $x = 40t$ and $y = 25 - 5t^2$. Take the origin at the base of the cliff. **2**

(ii) How far from the base of the cliff does the rock hit the sea? **2**

(b) Differentiate $\tan^{-1}\sqrt{x}$ with respect to *x*. **2**

(c) The function $f(x) = \log_e\left(3\sin x + 1\right)$ is defined over the domain $0 \le x \le \dfrac{\pi}{2}$.

(i) Find the inverse function $f^{-1}(x)$. **3**

(ii) What is the domain of the inverse function? **1**

(d) After *t* minutes, the rate of cooling of the temperature T $(°C)$ of a hot substance, when the surrounding temperature is *S*, is given by $\dfrac{dT}{dt} = -k(T - S)$ for some constant *k*.

(i) Show that the solution $T = S + Ae^{-kt}$, for some constant *A*, satisfies the differential equation $\dfrac{dT}{dt} = -k(T - S)$. **1**

(ii) Mrs Kuiters likes to drink hot water, but she will only drink it if it is between $60°C$ and $70°C$. Her kitchen is kept at a constant temperature of $25°C$. In her kitchen she pours a cup of water that boiled at $100°C$, and the water is exactly $10°C$ too warm to drink 4 minutes later. **4**

Calculate the maximum amount of time that she can spend enjoying her drink before it becomes too cold. Answer correct to the nearest second.

End of paper

ASSESSMENT TERM 2 2016

Mathematics Extension 1

Multiple Choice Answer Sheet Student Number: ____________________________

1	A ⬭	B ⬭	C ⬭	D ⬭
2	A ⬭	B ⬭	C ⬭	D ⬭
3	A ⬭	B ⬭	C ⬭	D ⬭
4	A ⬭	B ⬭	C ⬭	D ⬭
5	A ⬭	B ⬭	C ⬭	D ⬭

Year 12 Mathematics Extension 1 Solutions to T2 Assessment 2016

Section I – Multiple Choice

	A	B	C	D
1	○	○	●	○
2	●	○	○	○
3	○	○	●	○
4	○	●	○	○
5	○	○	●	○

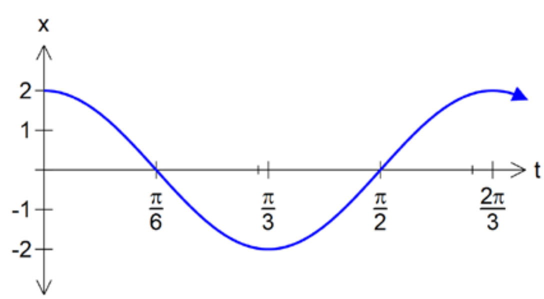
Section I – Multiple Choice Worked Solutions

1 $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ $=\sin\frac{2\pi}{3}$ $=\frac{\sqrt{3}}{2}$ (C)	2 $v^2=60+8x-4x^2$ $v^2=0: x^2-2x-15=0$ $(x-5)(x+3)=0$ $x=-3$ or 5 i.e. endpoints of motion $\therefore$ centre is $x=1$ (A)
3 $y=2\cos^{-1}(x-1)$ $-1\le x-1\le 1$ Domain: $0\le x\le 2$ Range: $0\le y\le 2\pi$ (C)	4 $v=\sqrt{9-x^2}$ $v^2=9-x^2$ SHM about centre O with $n=1$, $a=3$ $x=3\sin\left(t+\frac{\pi}{2}\right)$ is only particle with centre O and when $t=0$: $x=3$ (B)

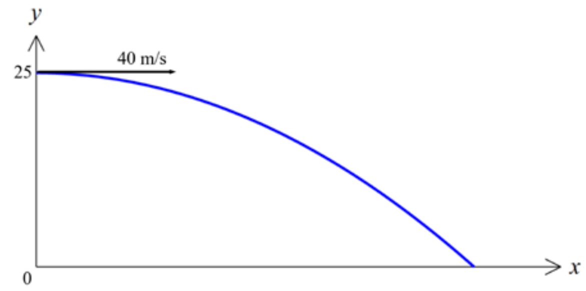
5	$\dot{y} = V\sin\alpha - gt$ $\dot{y} = 0 : t = \dfrac{V\sin\alpha}{g}$ $t = \dfrac{V\sin\alpha}{g}$: $y = V\sin\alpha\left(\dfrac{V\sin\alpha}{g}\right) - \dfrac{g}{2}\left(\dfrac{V\sin\alpha}{g}\right)^2$ $= \dfrac{V^2\sin^2\alpha}{g} - \dfrac{V^2\sin^2\alpha}{2g}$ $= \dfrac{2V^2\sin^2\alpha - V^2\sin^2\alpha}{2g}$ $= \dfrac{V^2\sin^2\alpha}{2g}$ (C)	

Section II – Working for Questions 6-7

Question 6		Mks	Marking Criteria
(a)(i)	$V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $160 = 4\pi r^2 \times \frac{dr}{dt}$ $\frac{dr}{dt} = \frac{160}{4\pi r^2}$ $\frac{dr}{dt} = \frac{40}{\pi r^2}$ mm/s	2	Correct solution **Careful when writing *V* and *r* as many letters were difficult to decipher**
		1	Correct chain rule with attempt at correct substitution into chain rule
(a)(ii)	$A = 4\pi r^2$ $\frac{dA}{dr} = 8\pi r$ $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $= 8\pi r \times \frac{40}{\pi r^2}$ $= \frac{320}{r}$ $r = 40 : \frac{dA}{dt} = \frac{320}{40}$ $= 8 \text{ mm}^2/\text{s}$	2	Correct solution
		1	Correct expression for $\frac{dA}{dt} = 8\pi r \times \frac{40}{\pi r^2}$ CFE

	Sample answer	Marks	Criteria
(b)	$\int \frac{1}{\sqrt{9-4x^2}}\,dx$ OR $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx$ $=\frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$ $\quad = \sin^{-1}\frac{f(x)}{a}+C$ $=\frac{1}{2}\sin^{-1}\frac{2x}{3}+C$ $\int \frac{1}{\sqrt{9-4x^2}}\,dx$ $=\frac{1}{2}\int \frac{2}{\sqrt{9-(2x)^2}}\,dx$ $=\frac{1}{2}\sin^{-1}\frac{2x}{3}+C$	**2**	Correct solution
		1	Correct attempt at the solution shown by $k\sin^{-1}\frac{2x}{3}$ or $\frac{1}{2}\sin^{-1}\frac{4x}{9}$
(c)	$n=3$ and $a=2$ as $v^2=9(4-x^2)$ is in the form $v^2=n^2(a^2-x^2)$ $t=0, v=0: x=\pm 2\cos 3t$ $T=\frac{2\pi}{3}$ Time taken to reach O $=\frac{1}{4}\left(\frac{2\pi}{3}\right)$ $=\frac{\pi}{6}$ OR $x=\pm 2\cos 3t$ $x=0:\ \pm 2\cos 3t=0$ $\cos 3t=0$ $3t=\frac{\pi}{2}$ $t=\frac{\pi}{6}$ 	**3**	Correct solution
		2	Correct values of a and n and correct period OR Substantially correct solution eg finds $x=(\pm)2\cos 3t$ or equivalent eg $2\sin\left(3t+\frac{\pi}{2}\right)$ OR Finds correct time from an incorrect trig equation
		1	Correct values of a and n OR Finds the correct expression for t: $t=\pm\frac{1}{3}\sin^{-1}\frac{x}{2}+C$

(d)(i)	$v = 10 - x$ $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= \frac{d}{dx}\left(\frac{(10-x)^2}{2}\right)$ $= \frac{-2(10-x)}{2}$ $= x - 10$	**2**	Correct solution
		1	Correct attempt at solution
(d)(ii)	$v = 10 - x$ $\frac{dx}{dt} = 10 - x$ $\frac{dt}{dx} = \frac{1}{10-x}$ $t = \int \frac{1}{10-x}\,dx$ $= -\ln\lvert 10-x\rvert + C$ $t = 0,\ x = 0:$ $0 = -\ln 10 + C$ $C = \ln 10$ $t = \ln 10 - \ln\lvert 10-x\rvert$ $t = \ln\left(\frac{10}{\lvert 10-x\rvert}\right)$ $\frac{10}{\lvert 10-x\rvert} = e^t$ $\lvert 10-x\rvert = 10e^{-t}$ $10 - x = 10e^{-t}$ or $-(10-x) = 10e^{-t}$ $t = 0$, $x = 0 : 10 - x = 10e^{-t}$ as $10e^{-t} > 0$ for all t $\therefore x = 10 - 10e^{-t}$	**3**	Correct solution
		2	Correct expression for t
		1	Correct attempt at the solution shown by stating $t = \int \frac{1}{10-x}\,dx$
(d)(iii)	$As\ t \to \infty,\ e^{-t} \to 0$ $\therefore x \to 10$ metres i.e. the limiting position of x is 10 metres	**1**	Correct answer

	Question 7	Mks	Marking Criteria
(a)(i)	 Initial conditions: $t=0,\ x=0,\ y=25$ $t=0,\ \dot{x}=40,\ \dot{y}=0$ Horizontally: $\ddot{x}=0$ $\dot{x}=C$ $t=0,\ \dot{x}=40: C=40$ $\dot{x}=40$ $x=40t+C_1$ $t=0,\ x=0: C_1=0$ $x=40t$ Vertically: $\ddot{y}=-10$ $\dot{y}=-10t+C_2$ $t=0,\ \dot{y}=0: C_2=0$ $\dot{y}=-10t$ $y=-5t^2+C_3$ $t=0,\ y=25: C_3=25$ $y=25-5t^2$ Markers Comment: The show needs to relate to the specific question asked. It is not asking you to reproduce the general formula but what happens when it is projected HORIZONTALLY in this case.	**2**	Correct solution
		1	Substantially correct solution
(a)(ii)	$y=0: 25-5t^2=0$ $5t^2=25$ $t^2=5$ $t=\sqrt{5}\ \ (t\geq 0)$ $t=\sqrt{5}: x=40\sqrt{5}$ $\therefore$ rock hits the sea $40\sqrt{5}$ metres from base of cliff	**2**	Correct solution
		1	Correct time or approach

(b)	$f(x) = \tan^{-1}\sqrt{x}$ $= \tan^{-1}\left(x^{\frac{1}{2}}\right)$ $f'(x) = \frac{1}{1+x} \times \frac{1}{2} x^{-\frac{1}{2}}$ $= \frac{1}{2(1+x)\sqrt{x}}$	**2**	Correct solution
		1	Correct attempt at chain rule OR Correct derivative of $\sqrt{x}$
(c)(i)	$f(x) = \ln(3\sin x + 1)$ for $0 \le x \le \frac{\pi}{2}$ Let $y = \ln(3\sin x + 1)$ Inverse function: $x = \ln(3\sin y + 1)$ $3\sin y + 1 = e^x$ $\sin y = \frac{e^x - 1}{3}$ $y = \sin^{-1}\left(\frac{e^x - 1}{3}\right)$	**3**	Correct solution
		2	Substantially correct solution
		1	Correct attempt at finding inverse function
(c)(ii)	Consider the range of $f(x)$: For $0 \le x \le \frac{\pi}{2}, 0 \le \sin x \le 1$ $0 \le 3\sin x \le 3$ $1 \le 3\sin x + 1 \le 4$ Range: $\ln 1 \le \ln(3\sin x + 1) \le \ln 4$ $\therefore 0 \le f(x) \le \ln 4$ Domain of $f^{-1}(x): 0 \le x \le \ln 4$	**1**	Correct answer
(d)(i)	$T = S + Ae^{-kt}$ $\frac{dT}{dt} = -k\left(Ae^{-kt}\right)$ $= -k(T - S)$ as $Ae^{-kt} = T - S$ $\therefore T = S + Ae^{-kt}$ is a solution of $\frac{dT}{dt} = -k(T - S)$	**1**	Correct solution

<table>
<tr>
<td rowspan="4">(d)(ii)</td>
<td rowspan="4">$T = S + Ae^{-kt}$
$S = 25 : T = 25 + Ae^{-kt}$
$t = 0, T = 100 :$
$100 = 25 + Ae^{0}$
$A = 75$
$T = 25 + 75e^{-kt}$
$t = 4, T = 80 :$
$80 = 25 + 75e^{-4k}$
$e^{-4k} = \frac{11}{15}$
$-4k = \ln\left(\frac{11}{15}\right)$
$k = -\frac{1}{4}\ln\left(\frac{11}{15}\right)$
$k = 0.077....$
$T = 60 : 60 = 25 + 75e^{-kt}$ where $k = 0.077...$
$e^{-kt} = \frac{7}{15}$
$-kt = \ln\left(\frac{7}{15}\right)$
$t = -\frac{1}{k}\ln\left(\frac{7}{15}\right)$
$t = 9.829....$
$T = 70 : 70 = 25 + 75e^{-kt}$ where $k = 0.077...$
$e^{-kt} = \frac{9}{15}$
$-kt = \ln\left(\frac{9}{15}\right)$
$t = -\frac{1}{k}\ln\left(\frac{9}{15}\right)$
$t = 6.588....$
Maximum amount of time to enjoy her drink
$= 9.829.... - 6.588....$
$= 3.241...$
$= 3\text{ min } 14\ \frac{28.13}{60}\text{ sec}$
$= 3\text{ min } 14\text{ sec}$ (nearest sec)</td>
<td>4</td>
<td>Correct solution

Please do not round until the last calculation</td>
</tr>
<tr>
<td>3</td>
<td>Find time taken for temperature to reach either $60°C$ or $70°C$</td>
</tr>
<tr>
<td>2</td>
<td>Find the values of k</td>
</tr>
<tr>
<td>1</td>
<td>Find the values of S and A</td>
</tr>
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